$$(1.2.3.....a)^2+(-1)^a \equiv 0 \pmod{2a+1} \qquad (12).$$

$$\therefore (1.2.3.....a)^4-1 \equiv 0 \pmod{2a+1} \qquad (13).$$

Since $a+1$ and $2a+1$ are prime to each other, (11) and (13) lead to the theorem stated.

PROPOSITION E. *If* $(1.2.3.....a)^4 \equiv 0$ [*mod* $(a+1)(2a+1)$], $a+1$ and $2a+1$ *are both prime.*

We have to show that $a+1$ and $2a+1$ are prime when (11) and (13) hold.

If $a+1=4$, (11) does not hold. For all other values of $a+1$ *not* prime, $1.2.3.....a \equiv 0 \pmod{a+1}$. Hence, when $a+1$ is not prime, (11) does not hold. Therefore, $a+1$ is prime under the given condition.

If $2a+1$ is *not* prime, $1.2.3.....2a \equiv 0 \pmod{2a+1}$.

$$\therefore (1.2.3.....(a-1)a.[(2a+1)-a][(2a+1)-(a+1)].....$$
$$.....[(2a+1)-2][(2a+1)-1] \equiv 0 \pmod{2a+1}.$$

Expanding and casting out terms containing $2a+1$ as a factor, we get

$$(1.2.3.....a)^2 \equiv 0 \pmod{2a+1} \qquad (14),$$

if $2a+1$ is *not* prime. Hence, when (13) holds, as (14) does not then hold, $2a+1$ is odd.

PROPOSITION F. *If* $a+1$ *and* $2a+1$ *are both prime,* $(a+1)(2a+1)$ *may be expressed as the sum of* r *and* s *such that* $r^4-1 \equiv 0$, $s^4-1 \equiv 0$, *and* $r^2s^2-1 \equiv 0$ [*mod* $(a+1)(2a+1)$].

The demonstration depends upon Proposition D above, and is similar in method to that of Proposition C. It may easily be supplied by the reader.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

No. 223 was also solved by L. E. Newcomb.

Mr. L. S. Shively calls attention to the fact that a solution of No. 225 is given in C. Smith's "A Treatise on Algebra," page 183, Ex. 4.

226. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Find the real roots of the system

$$x^2+w^2+v^2=a^2,\quad vw+u(y+z)=bc,$$
$$w^2+y^2+u^2=b^2,\quad wu+v(z+x)=ca,$$
$$v^2+u^2+z^2=c^2,\quad uv+w(x+y)=ab.$$

Solution by A. H. HOLMES, Brunswick, Me.

$$x^2+w^2+v^2=a^2 \ldots\ldots(1),\quad vw+u(y+z)=bc\ldots\ldots(4),$$
$$w^2+y^2+u^2=b^2\ldots\ldots(2),\quad wu+v(z+x)=ca\ldots\ldots(5),$$
$$v^2+u^2+z^2=a^2\ldots\ldots(3),\quad uv+w(x+y)=ab\ldots\ldots(6).$$

From (2), (3), and (4),

$$(w^2+y^2+u^2)(v^2+u^2+z^2)=[vw+u(y+z)]^2.$$
$$\therefore (yv-uw)^2+(uv-zw)^2+(u^2-yz)^2=0.$$
$$\therefore yv=uw\ldots\ldots(7);\ uv=zw\ldots\ldots(8);\ u^2=yz\ldots\ldots(9).$$

From (1), (3), and (5),

$$(x^2+w^2+v^2)(v^2+u^2+z^2)=[wu+v(z+x)]^2.$$
$$\therefore v^2=xz\ldots\ldots(10);\ vw=xu\ldots\ldots(11).$$

From (2), (3), and (6),

$$(w^2+y^2+u^2)(v^2+u^2+z^2)=[uv+w(x+y)]^2.$$
$$\therefore w^2=xy\ldots\ldots(12).$$

Substituting xy and xz for w^2 and v^2 in (1), xy and yz for w^2 and u^2 in (2), and xz and yz for v^2 and u^2 in (3), adding the three equations and extracting square root, we obtain $x+y+z=\sqrt{(a^2+b^2+c^2)}$.

Substituting xu for vw in (4), $u=\dfrac{bc}{\sqrt{(a^2+b^2+c^2)}}$.

Similarly, $v=\dfrac{ac}{\sqrt{(a^2+b^2+c^2)}}$, $w=\dfrac{ab}{\sqrt{(a^2+b^2+c^2)}}$,

$$x=\frac{a^2}{\sqrt{(a^2+b^2+c^2)}},\ y=\frac{b^2}{\sqrt{(a^2+b^2+c^2)}},\ z=\frac{c^2}{\sqrt{(a^2+b^2+c^2)}}.$$

227. Proposed by G. I. HOPKINS, A. M., Manchester, N. H.

Solve $x+y+xy+x^2y+xy^2+x^3y+2x^2y^2+xy^3+x^3y^2+x^2y^3=11$; $x^4y+3x^3y^2+3x^2y^3+2x^4y^2+4x^2y^3+2x^2y^4+4x^4y^3+4x^3y^4+xy^4+x^5y^2+x^5y^3+2x^4y^4+x^2y^5+x^3y^5=30$.

Solution by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

Put $X=(x+y+xy+x^2y+xy^2)$, and $Y=(x^3y+2x^2y^2+xy^3+x^3y^2+x^2y^3)$; then the given equations become, respectively,

$$X+Y=11\ldots\ldots(\alpha),\text{ and}$$
$$XY=30\ldots\ldots(\beta).$$

$\therefore X=6$, or 5; and $Y=5$, or 6.